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A quick proof on the equivalence classes of extended Vogan diagrams ☆

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Abstract

An extended Vogan diagram is an extended Dynkin diagram together with a diagram involution, such that the vertices fixed by the involution are colored white or black. Every extended Vogan diagram represents an almost compact real form of the affine Kac–Moody Lie algebra. Two extended diagrams are said to be equivalent if they represent isomorphic real forms. The equivalence classes of extended Vogan diagrams have earlier been classified by the authors. In this paper, we present a much shorter and instructive argument.

Keywords: Dynkin diagram; Extended Vogan diagram; Simple Lie algebra

1. Introduction

Let X be a Dynkin diagram. Let $X^{(1)}$, $A_n^{(2)}$, $D_n^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$ be the extended or affine diagram [6, Chapter X-5]. For example if $X = A_n$, which is a line with n vertices and $n-1$ edges, then $X^{(1)}$ is a loop with $n+1$ vertices. A Vogan diagram on X or $X^{(k)}$ is a diagram involution, such that the vertices fixed by the involution are colored white or black. Every Vogan diagram on X represents a real simple Lie algebra, and every Vogan diagram on $X^{(k)}$ represents an almost compact real form of the affine Kac–Moody Lie algebra [1–3]. If two diagrams represent isomorphic Lie algebras, we say that they are equivalent. Thus the classification of the Lie algebras amounts to the classification of equivalence classes of Vogan diagrams on X [4] and on $X^{(k)}$ [5]. Some

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combinatorial arguments for $X^{(1)}$ in [5] are messy and not instructive. For example, the lengthy arguments [5, pp. 138–147] merely show that two particular pairs of diagrams in $E_7^{(1)}$ and $E_8^{(1)}$ are not equivalent. This paper provides a much shorter and instructive proof on the equivalence classes of $X^{(1)}$, thereby greatly simplifies the arguments in [5].

2. Induced diagrams

Let $V(X)$ and $V(X^{(1)})$ be the Vogan diagrams of X and $X^{(1)}$ with trivial diagram involution. Given any labeling $1,\dots,n$ on the vertices of X, we let $1,\dots,n,e$ be the corresponding labeling on the vertices of $X^{(1)}$, where e is the extra vertex in $X^{(1)}$. An element of $V(X)$ or $V(X^{(1)})$ is denoted by $(i_1,\dots,i_k)$, where $i_1,\dots,i_k$ are the black vertices. Each vertex i of $X^{(1)}$ is assigned a positive integer m_i [6, p. 503]. Define

$$\begin{aligned} &I : V(X) \longrightarrow V\big(X^{(1)}\big), \\ &I(i_1,\dots,i_k) = \begin{cases} (i_1,\dots,i_k) & \text{if } m_{i_1}+\dots+m_{i_k} \text{ is even;} \\ (i_1,\dots,i_k,e) & \text{if } m_{i_1}+\dots+m_{i_k} \text{ is odd.} \end{cases} \end{aligned} \tag{2.1}$$

We call $I(v) \in V(X^{(1)})$ the *induced diagram* of $v \in V(X)$. An element of $V(X^{(1)})$ not belonging to the image of I is said to be *noninduced*.

For example, label the vertices of C_n by $1,2,\dots,n$, where n is the unique long root. Here $m_1=\dots=m_{n-1}=2$ and $m_n=1$. Consider $(1,2)\in V(C_n)$, namely the Vogan diagram with vertices 1 and 2 colored black. We have $I(1,2)=(1,2)$ because $m_1+m_2=4$ is even. Similarly, $I(1,n)=(1,n,e)$ because $m_1+m_n=3$ is odd.

Theorem 1. *Two Vogan diagrams $v,w\in V(X)$ are equivalent if and only if their induced diagrams $I(v),I(w)\in V(X^{(1)})$ are equivalent.*

Proof. A way to view the coefficients m_i is as follows: The vertices of X represent the simple roots Π of a finite dimensional complex simple Lie algebra, and the extra vertex e represents the lowest root. The coefficients m_i are introduced by Kac while he studies finite order automorphisms [7]. Namely the coefficients of Π are the coefficients of the highest root with respect to Π, and e has coefficient 1. Hence the linear combination of $\Pi\cup\{e\}$ over the coefficients $\{m_i\}$ is 0.

Recall that the white (respectively black) vertices of a Vogan diagram represent the compact (respectively noncompact) roots of a real simple Lie algebra; namely their root spaces are in the 1 (respectively -1) eigenspace of a Cartan involution of the Lie algebra. If α, β and $\alpha+\beta$ are roots, then the compactness (i.e. whether compact or noncompact) of $\alpha,\beta,\alpha+\beta$ are related by the law

$$c+c=c, \qquad c+n=n, \qquad n+n=c, \tag{2.2}$$

where c denotes compact root and n denotes noncompact root. In this way, the compactness of e is determined by the compactness of the simple roots. The vertex e of an induced diagram $I(i_1,\dots,i_k)$ satisfies

$$\begin{aligned} e \text{ is white} \quad &\Longleftrightarrow \quad m_{i_1}+\dots+m_{i_k} \text{ is even} \quad \text{by (2.1)}, \\ &\Longleftrightarrow \quad e \text{ is compact} \quad \text{by (2.2)}. \end{aligned}$$

Hence in an induced diagram in $V(X^{(1)})$, the vertices receive an "over-determined" coloring, where the color of e correctly represents its compactness. Similarly, in a noninduced diagram, the color of e does not correctly represent its compactness.

We conclude that every induced diagram $I(v)$ represents a finite dimensional real simple Lie algebra, as given by v. The equivalence relation, as defined by the algorithms F_i in [4, (2.1)] and [5, (1.2)], satisfies $v \sim w$ if and only if $I(v) \sim I(w)$. This proves the theorem. □

3. Examples

With the aid of Theorem 1, we can use the equivalence classes of $V(X)$ [4] to solve many equivalence classes of $V(X^{(1)})$ [5]. It greatly reduces the messy computations in [5]. We illustrate this with the following two examples.

We have used lengthy computations in [5, pp. 138–147] to show that the two $E_7^{(1)}$ diagrams in Fig. 1 are not equivalent,

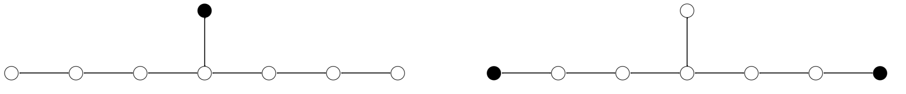

Fig. 1.

and that the two $E_8^{(1)}$ diagrams in Fig. 2 are not equivalent.

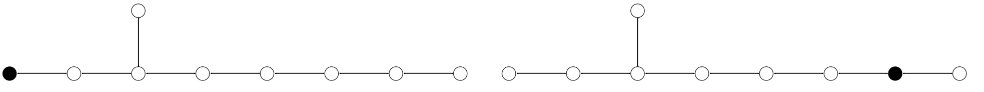

Fig. 2.

Theorem 1 helps to simplify their arguments. By the coefficients of the roots [6, p. 503], we see that all diagrams in Figs. 1 and 2 are induced diagrams. Namely, I of (2.1) maps the following E_7 diagrams

Fig. 3.

to the diagrams in Fig. 1. The diagrams in Fig. 3 are not equivalent [4]. Consequently, by Theorem 1, the diagrams in Fig. 1 are also not equivalent.

Similarly, I maps the following E_8 diagrams

Fig. 4.

to the diagrams in Fig. 2. The diagrams in Fig. 4 are not equivalent [4], so by Theorem 1, the diagrams in Fig. 2 are also not equivalent.

More generally, consider [5, Table 1]. For each $V(X^{(1)})$, the equivalence classes of the induced diagrams can be checked by Theorem 1 and [4, Table 1].

The above technique does not cover the noninduced diagrams, or diagrams with nontrivial involution, or diagrams of $A_n^{(2)}$, $D_n^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$ in [5]. However, the arguments in [5] for these diagrams are quite straight forward. The more messy arguments occur in the induced diagrams of $V(X^{(1)})$, and they are handled by the above technique.

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